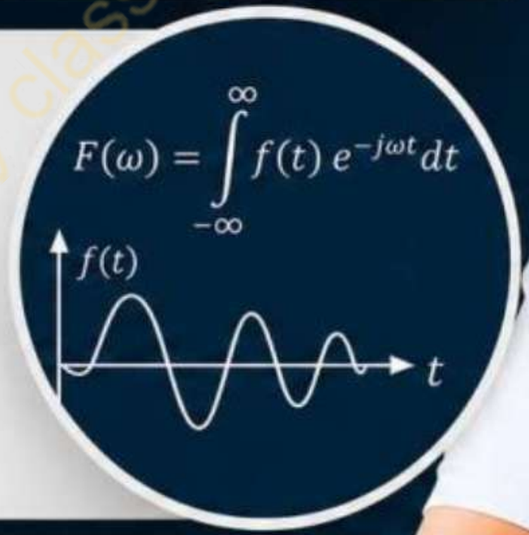
SEMESTER EXAM PREPARATION

FOR B.TECH.

ELECTRICAL & ALLIED BRANCH

TOPIC TO BE COVERED :-

Fourier Transform
and its properties



COMPLETE THEORY | IMPORTANT PROPERTIES | EXAM FOCUSED | HIGH SCORING TOPIC

Avinash Sir

Fourier transform

The Fourier Transform (FT) is a mathematical tool used to convert a time-domain signal into its frequency-domain representation. It tells us which frequencies are present in a signal and their amplitudes.

Why Fourier Transform?

Converts time domain → frequency domain

Simplifies analysis of Linear Time Invariant (LTI) systems

Used in communication systems

Used in digital signal processing

Used in image processing

Used in control systems

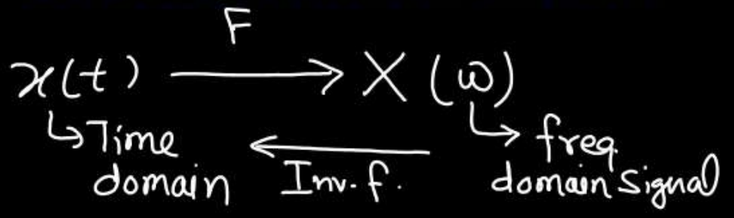
Used in spectrum analysis

Gateway classes

Continuous-Time Fourier Transform (CTFT)

The Fourier transform of a continuous-time function x(t) is defined as,

Forward Fourier Transform



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Inverse Fourier Transform; The inverse Fourier transform of a continuous-time function is defined as,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Gatewayclasses

where

- $x(t)$= Time-domain signal
- $X(\omega)$= Frequency-domain signal
- ω= Angular frequency (rad/sec)

Difference Between CTFT and DTFT

Continuous-Time Fourier Transform (CTFT)	Discrete-Time Fourier Transform (DTFT)
Applied to **continuous-time signals** $x(t)$.	Applied to **discrete-time signals** $x[n]$.
Time variable is **continuous** (t).	Time variable is **discrete** (n).
Frequency variable is **continuous** and **non-periodic** (ω).	Frequency variable is **continuous** but **periodic** with period 2π.
Forward Transform: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$	Forward Transform: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$
Inverse Transform: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$	Inverse Transform: $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$
Spectrum is **continuous and non-periodic**.	Spectrum is **continuous and periodic** with period 2π.
Mainly used for **analog (continuous-time)** signals.	Mainly used for **digital (discrete-time)** signals.
Integration is used in the transform.	Summation is used in the transform.
Common applications: Analog communication, analog filters.	Common applications: DSP, digital filters, speech and image processing.

Gateway Classes

Fourier Transform from Fourier Series

Any periodic signal can be represented as a sum of complex exponential signals using the Complex Exponential Fourier Series. When the period T approaches infinity, the periodic signal becomes an aperiodic signal. In this limiting process, the Fourier Series transforms into the Continuous-Time Fourier Transform (CTFT).

A periodic signal $x(t)$ of period T can be represented as

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Where, C_n = Fourier Series coefficient , $\omega_0 = \frac{2\pi}{T}$ = Fundamental angular frequency

The Fourier Series coefficient is

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}dt$$

Choosing one complete period symmetrically, $t_0 = -\frac{T}{2}$

Therefore,

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}dt$$

Gateway classes

Fourier Transform from Fourier Series

Replace $n\omega_0$ by ω

Let

$$\boxed{\omega = n\omega_0}$$

Then

$$\boxed{TC_n = \int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt}$$

Apply the Limit $T \to \infty$

For an **aperiodic signal**, the period becomes infinitely large.

Hence, $T \to \infty$

And $-\frac{T}{2} \to -\infty, \frac{T}{2} \to +\infty$

Therefore,

$$\boxed{\lim_{T\to\infty} (TC_n) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$$

The above integral depends only on the frequency variable ω. Hence, it is defined as the **Fourier Transform** of the signal $x(t)$.

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$$

Gateway classes

2 Marks Imp

(CTFT)

Convergence of Fourier Transform

Fourier transform converges of it satisfies following ***Dirichlet conditions***. X(t) is a continuous time signal. The conditions are:

1. x(t) should be single valued in any finite time interval T.
2. It should have a finite number of discontinuities in any finite interval T.
3. x(t) has a finite number of maxima and minima in any time period T.
4. x(t) should be absolutely integrable function. This means that

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

Gateway classes

Convergence of DTFT

Existence of DTFT

The Discrete-Time Fourier Transform (DTFT) of a discrete-time signal $x[n]$is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Existence Condition

The DTFT exists if the sequence is absolutely summable.

That is,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

This is called the absolute summability condition.

Gateway classes

Properties of Fourier Series

$\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

1. Linearity Property

Statement

If

$$x_1(t) \overset{FT}{\Leftrightarrow} X_1(\omega)$$

and

$$x_2(t) \overset{FT}{\Leftrightarrow} X_2(\omega)$$

then

$$ax_1(t) + bx_2(t) \overset{FT}{\Leftrightarrow} aX_1(\omega) + bX_2(\omega)$$

Proof

Let

$$y(t) = ax_1(t) + bx_2(t)$$

Taking Fourier Transform,

$$Y(\omega) = \int_{-\infty}^{\infty} (ax_1(t) + bx_2(t))e^{-j\omega t} dt$$

Using linearity of integration,

$$= a\underbrace{\int x_1(t)e^{-j\omega t} dt}_{X_1(\omega)} + b\underbrace{\int x_2(t)e^{-j\omega t} dt}_{X_2(\omega)}$$

Since

$$\int x_1(t)e^{-j\omega t} dt = X_1(\omega)$$

and

$$\int x_2(t)e^{-j\omega t} dt = X_2(\omega)$$

Therefore

$$Y(\omega) = aX_1(\omega) + bX_2(\omega)$$

Hence proved.

Gateway classes

Properties of Fourier Series

2. Time Shifting Property

$x(t) \longrightarrow X(\omega)$

Statement

$$x(t - t_0) \Leftrightarrow e^{-j\omega t_0} X(\omega)$$

Proof

Let

$$y(t) = x(t - t_0)$$

Taking FT,

$$Y(\omega) = \int x(t - t_0) e^{-j\omega t} dt$$

Substitute $u = t - t_0$

Then $t = u + t_0$

and

$$dt = du$$

Hence

$e^{-j\omega u} \cdot e^{-j\omega t_0}$

$$Y(\omega) = \int x(u) e^{-j\omega(u+t_0)} du$$

$$= e^{-j\omega t_0} \int x(u) e^{-j\omega u} du$$

$$= e^{-j\omega t_0} X(\omega)$$

Hence proved.

Gateway classes

Properties of Fourier Series

3. Frequency Shifting Property

Statement

$$e^{j\omega_0 t}x(t) \Leftrightarrow X(\omega - \omega_0)$$

Proof

Let

$$y(t) = e^{j\omega_0 t}x(t)$$

Then

$$Y(\omega) = \int x(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

Combine exponents,

$$= \int x(t)e^{-j(\omega-\omega_0)t}dt$$

Recognize FT,

$$Y(\omega) = X(\omega - \omega_0)$$

Hence proved.

Duality or Symmetry Property

Statement

If

$$x(t) \Leftrightarrow X(\omega)$$

then

$$X(t) \overset{FT}{\Leftrightarrow} 2\pi x(-\omega)$$

Proof

Starting from the inverse transform,

$$x(t) = \frac{1}{2\pi}\int X(\omega)e^{j\omega t}d\omega$$

Replace

$$t \to -\omega$$

and interchange the variables t and ω. Rearranging gives

$$\int X(t)e^{-j\omega t}dt = 2\pi x(-\omega)$$

Therefore,

$$X(t) \Leftrightarrow 2\pi x(-\omega)$$

Hence proved.

Gateway classes

Properties of Fourier Series

4. Time Scaling Property

Statement

$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Proof

Let

$$y(t) = x(at)$$

Taking FT,

$$Y(\omega) = \int x(at) e^{-j\omega t} dt$$

Substitute $u = at$

Then

$$dt = \frac{du}{a}$$

If $a > 0$, +ve

then

$$Y(\omega) = \frac{1}{a} \int x(u) e^{-j(\omega/a)u} du$$

$X\left(\frac{\omega}{a}\right)$

Recognize FT,

$$Y(\omega) = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

For negative a, the limits reverse, introducing the absolute value:

$$Y(\omega) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Hence proved.

Gateway classes

Properties of Fourier Series

Imp

Differentiation Property (Time Domain)

Property

If

$$x(t) \overset{\mathcal{F}}{\leftrightarrow} X(\omega)$$

then

$$\boxed{\frac{d}{dt}x(t) \overset{\mathcal{F}}{\leftrightarrow} j\omega X(\omega)}$$

Proof

From the inverse Fourier Transform,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$

Differentiate both sides with respect to t,

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\frac{d}{dt}\left[\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega\right]$$

Since differentiation can be moved inside the integral,

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}\left(e^{j\omega t}\right) d\omega$$

Now,

$$\frac{d}{dt}\left(e^{j\omega t}\right) = j\omega e^{j\omega t}$$

Substituting,

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)(j\omega)e^{j\omega t}\, d\omega$$

Rearranging,

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(j\omega X(\omega)\right)e^{j\omega t}\, d\omega$$

Comparing with the inverse Fourier Transform,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$

it follows that

$$\boxed{\mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = j\omega X(\omega)}$$

Hence,

$$\boxed{\frac{d}{dt}x(t) \overset{\mathcal{F}}{\leftrightarrow} j\omega X(\omega)}$$

Hence Proved.

Gateway classes

Properties of Fourier Series

Imp

6. Convolution Property

Statement

$$x_1(t) * x_2(t) \Leftrightarrow X_1(\omega)X_2(\omega)$$

Proof

Let

$$y(t) = x_1(t) * x_2(t) = \int x_1(\tau)x_2(t-\tau)d\tau$$

Taking FT,

$$Y(\omega) = \int_{-\infty}^{\infty}\left[\int x_1(\tau)x_2(t-\tau)d\tau\right]e^{-j\omega t}dt$$

Interchange integrals,

$$= \int x_1(\tau)\left[\int x_2(t-\tau)e^{-j\omega t}dt\right]d\tau$$

Time shifting

Substitute

$$u = t - \tau$$

Then

$$t = u + \tau$$

Hence

$$= \int x_1(\tau)e^{-j\omega\tau}\left[\int x_2(u)e^{-j\omega u}du\right]d\tau$$

Recognize

$$= X_2(\omega)\int x_1(\tau)e^{-j\omega\tau}d\tau$$

$$= X_1(\omega)X_2(\omega)$$

Hence proved.

Gateway classes

Properties of Fourier Series

Parseval's Theorem

$[x(t) * x^*(-t)] \rightarrow |x(t)|^2$

Statement

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

Proof

Using the inverse Fourier transform,

$$x(t) = \frac{1}{2\pi} \int X(\omega) e^{j\omega t} d\omega$$

Multiply by $x^*(t)$ and integrate over all time:

$$\int x(t) x^*(t) \, dt = \frac{1}{2\pi} \int X(\omega) \underbrace{\left[\int x^*(t) e^{j\omega t} dt \right]}_{X^*(\omega)} d\omega$$

Since

$$\int x^*(t) e^{j\omega t} dt = X^*(\omega),$$

we obtain

$$\int |x(t)|^2 \, dt = \frac{1}{2\pi} \int |X(\omega)|^2 \, d\omega$$

Hence proved.

Gateway classes

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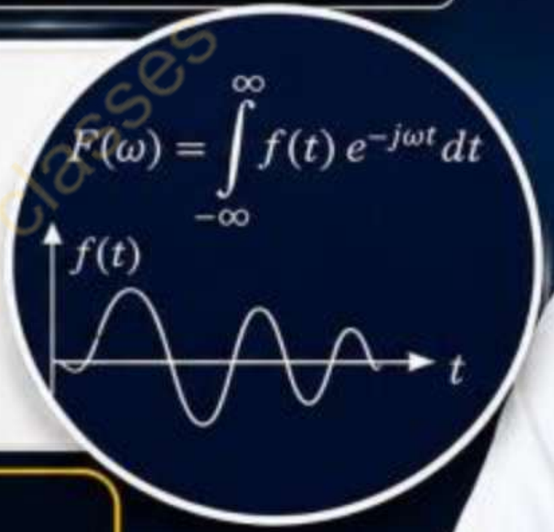
Fourier Transform



Gateway Classes

TOPICS TO BE COVERED :-

- Numerical Problems on Fourier Transform
- Applications of Fourier Transformer

COMPLETE THEORY | IMPORTANT PROPERTIES | EXAM FOCUSED | HIGH SCORING TOPIC

Avinash Sir

Fourier Transform of basic Functions

Ex.1 Determine the Fourier Transform of Unit Impulse Signal $x(t) = \delta(t)$

Sol. From the definition of Fourier Transform,

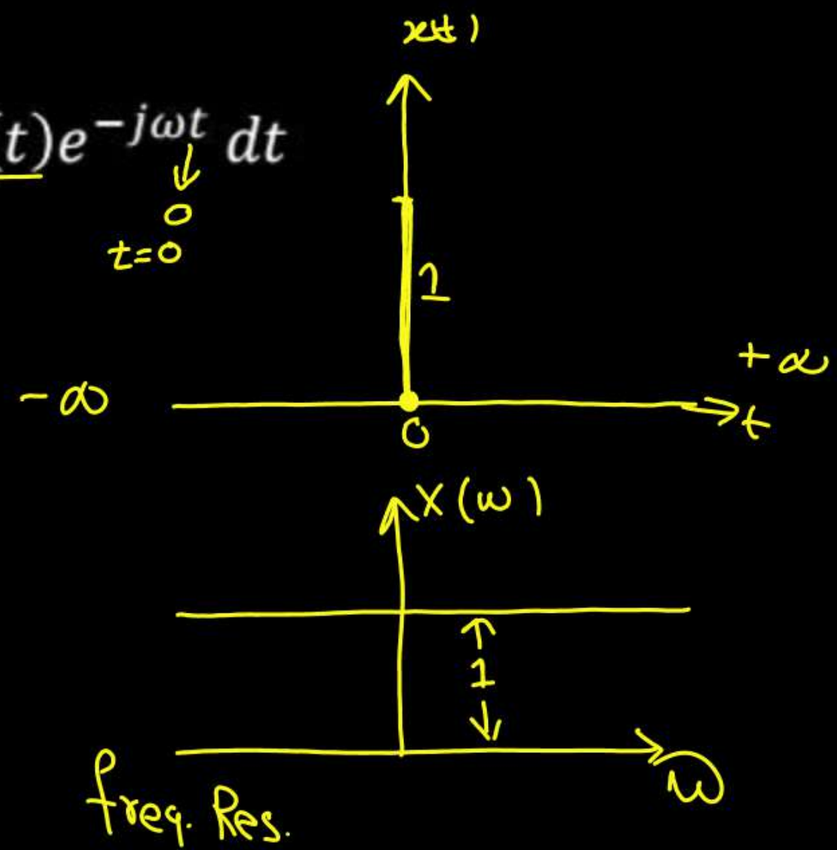
$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt$$

Apply the property of impulse signal, $\delta(t)y(t) = \delta(t)y(0)$

$$\Rightarrow FT[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{0}\,dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\cdot 1\,dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\,dt$$



Gateway classes

The area under the unit impulse signal $\delta(t)$ is **1**.

$$\boxed{FT[\delta(t)] = X(\omega) = 1}$$

Ex.2 Determine the Fourier Transform of decaying exponential signal $x(t) = e^{-at}u(t),\ a > 0$. Hence obtain its magnitude and phase spectrum.

Sol. From the definition of Fourier Transform,

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$\Rightarrow FT[e^{-at}u(t)] = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt$$

Since, $u(t) = 1, t > 0$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{e^{-\infty} - e^{0}}{-(a+j\omega)}$$

$$= \frac{0-1}{-(a+j\omega)} = \frac{1}{a+j\omega}$$

Hence,

$$\boxed{FT[e^{-at}u(t)] = X(\omega) = \frac{1}{a+j\omega}}$$

Since the frequency-domain function is complex, its magnitude and phase spectra are:

Magnitude Spectrum

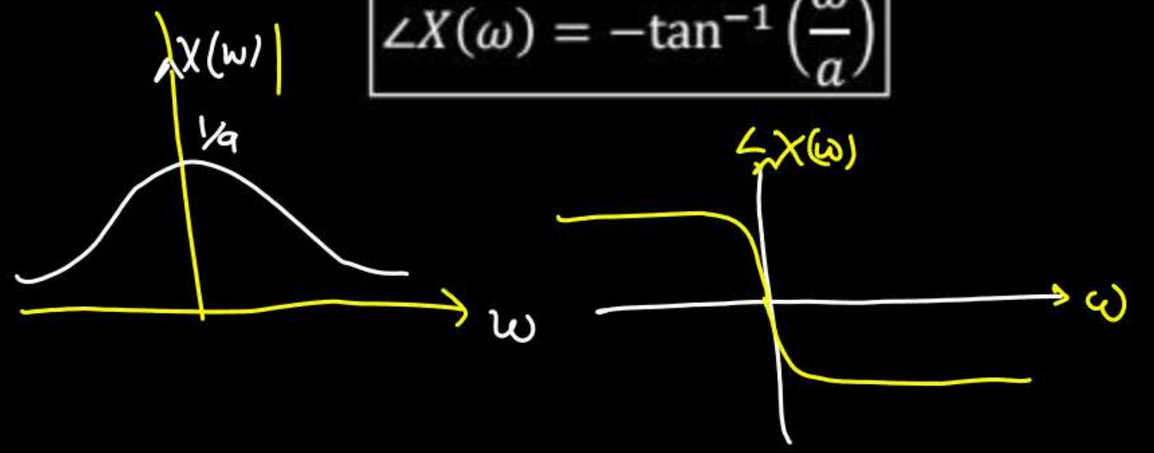
$$\boxed{|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}}$$

Phase Spectrum

$$\angle X(\omega) = \angle(1+j0) - \angle(a+j\omega)$$

$$= 0 - \tan^{-1}\left(\frac{\omega}{a}\right)$$

$$\boxed{\angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)}$$



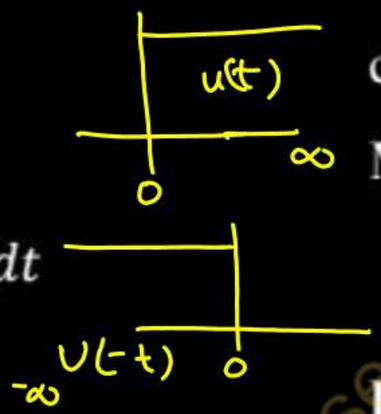
Gateway classes

Ex.3 Determine the Fourier Transform of decaying exponential signal $x(t) = e^{at}u(-t),\ a > 0$. Hence obtain its magnitude and phase spectrum.

Sol. From the definition of Fourier Transform,

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$\Rightarrow FT[e^{-at}u(t)] = \int_{-\infty}^{\infty} e^{at}\,u(-t)e^{-j\omega t}\,dt$$



Since, $u(-t) = 1, t < 0$

$$= \int_{-\infty}^{0} e^{-(a-j\omega)t}\,dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{-\infty}^{0} = \frac{e^{0}-e^{-\infty}}{(a-j\omega)}$$

$$= \frac{1-0}{(a-j\omega)} = \frac{1}{a-j\omega}$$

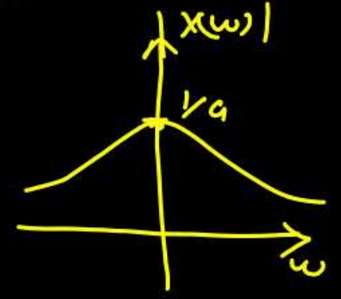
Hence,

$$\boxed{FT[e^{at}u(-t)] = X(\omega) = \frac{1}{a-j\omega}}$$

Since the frequency-domain function is complex, its magnitude and phase spectra are:

Magnitude Spectrum

$$\boxed{|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}}$$

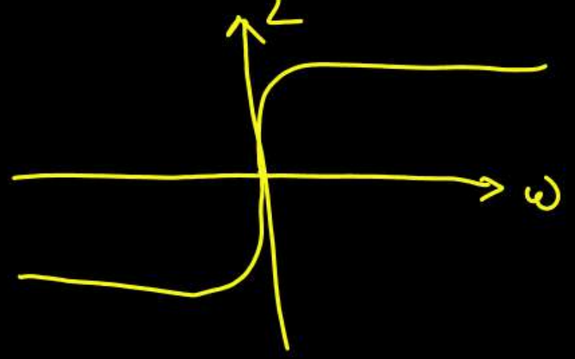


Phase Spectrum

$$\angle X(\omega) = \angle(1+j0) - \angle(a-j\omega)$$

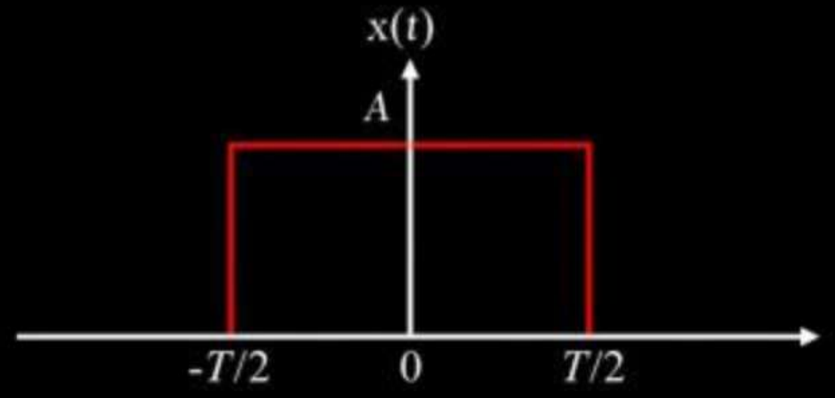
$$= 0 - \tan^{-1}\left(\frac{-\omega}{a}\right)$$

$$\boxed{\angle X(\omega) = \tan^{-1}\left(\frac{\omega}{a}\right)}$$



Gateway classes

Ex.4 Determine the Fourier Transform of signal $x(t)$



Sol. $x(t) = \begin{cases} A & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$

$$F[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{T/2} A e^{-j\omega t}\, dt$$

$$= A \left[\frac{e^{-j\omega t}}{-j\omega} \right]_{-T/2}^{+T/2}$$

$$= A \left[\frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega} \right]$$

$$= \frac{A}{j\omega} \left[e^{j\omega T/2} - e^{-j\omega T/2} \right]$$

$$= \frac{A}{j\omega} \left[2j \sin\frac{\omega T}{2} \right]$$

$$X(\omega) = \frac{A}{\omega} \left[2 \sin\left(\frac{\omega T}{2}\right) \right]$$

$$= 2A \left(\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\omega \cdot \frac{T}{2}\right)} \right) \frac{T}{2}$$

$$X(\omega) = 2T \operatorname{Sinc}\left(\frac{\omega T}{2}\right)$$

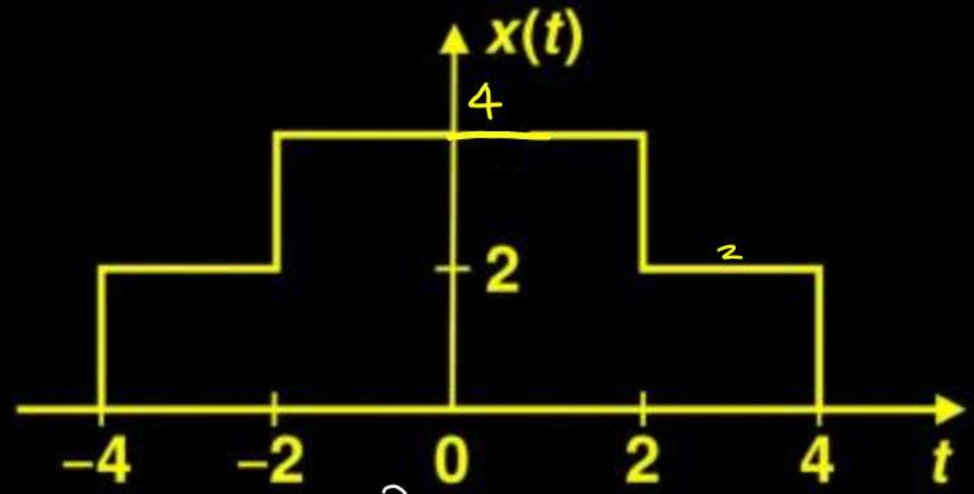
$$\frac{e^{+j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

$$e^{+j\theta} - e^{-j\theta} = 2j\sin\theta$$

Gateway classes

Ex. 6 Determine the Fourier Transform of signal $x(t)$

Sol. $$x(t) = \begin{cases} 2 & -4 < t < -2 \\ 4 & -2 < t < 2 \\ 2 & 2 < t < 4 \end{cases}$$



$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-4}^{-2} 2e^{-j\omega t} dt + \int_{-2}^{2} 4e^{-j\omega t} dt + \int_{2}^{4} 2e^{-j\omega t} dt$$

$$= 2\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-4}^{-2} + 4\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-2}^{2} + 2\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{2}^{4}$$

$$= 2\left[\frac{e^{+2j\omega} - e^{4j\omega}}{-j\omega}\right] + 4\left[\frac{e^{-2j\omega} - e^{2j\omega}}{-j\omega}\right] + 2\left[\frac{e^{-4j\omega} - e^{-2j\omega}}{-j\omega}\right]$$

$$= \frac{-2}{-j\omega}\left[e^{2j\omega} - e^{4j\omega} + 2e^{-2j\omega} - 2e^{2j\omega} + e^{-4j\omega} - e^{-2j\omega}\right]$$

$$= \frac{-2}{j\omega}\left[-e^{2j\omega} + e^{-2j\omega}\right) + \left(e^{-4j\omega} - e^{4j\omega}\right)\Big]$$

$$= \frac{-2}{j\omega}\left[-\left(e^{2j\omega} - e^{-2j\omega}\right) - \left(e^{4j\omega} - e^{-4j\omega}\right)\right]$$

$$= \frac{2}{j\omega}\left[2j\sin(2\omega) + 2j\sin(4\omega)\right]$$

$$= \frac{4}{\omega}\left[\sin 2\omega + \sin 4\omega\right]$$

Ans.

Gateway classes

Ex. 7 Find the Fourier Transform of the signal

$$x(t) = \begin{cases} e^{-|t|}, & \text{for } -2 \le t \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Sol. Fourier Transform (CTFT) is

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{-2}^{2} e^{-|t|} e^{-j\omega t}\, dt$$

$t \to -t$ $\qquad$ $t \to +t$

$$X(\omega) = \int_{-2}^{0} e^{t} e^{-j\omega t} dt + \int_{0}^{2} e^{-t} e^{-j\omega t} dt$$

$$= \int_{-2}^{0} e^{(1-j\omega)t}\, dt + \int_{0}^{2} e^{-(1+j\omega)t}\, dt$$

$$= \left[\frac{e^{(1-j\omega)t}}{1-j\omega}\right]_{-2}^{0} + \left[\frac{-e^{-(1+j\omega)t}}{1+j\omega}\right]_{0}^{2}$$

$$= \frac{1-e^{-2(1-j\omega)}}{1-j\omega} + \frac{1-e^{-2(1+j\omega)}}{1+j\omega}$$

at $t \to +t$

$e^{-t} \to 0-2$

at $t = -t$

$e^{t} = -2 \to 0$

$-2 \quad 0 \quad 2$

$$\boxed{X(\omega) = \frac{1-e^{-2(1-j\omega)}}{1-j\omega} + \frac{1-e^{-2(1+j\omega)}}{1+j\omega}}$$

Ans

Using

$$e^{-2(1\pm j\omega)} = e^{-2} e^{\mp j2\omega}$$

and

$$e^{j2\omega} + e^{-j2\omega} = 2\cos 2\omega,$$

after simplification,

$$\boxed{X(\omega) = \frac{2}{1+\omega^2}\left[1 - e^{-2}\cos(2\omega) + \omega e^{-2}\sin(2\omega)\right]}$$

Gateway classes

Ex. 8 Find the Fourier Transform of the signal

(Same as Ex. 4)

$$x(t) = \begin{cases} A, & |t| < T_0 \\ 0, & |t| > T_0 \end{cases}$$

$-t < T_0 \quad t < -T_0$
$t < T_0 \quad t < T_0$

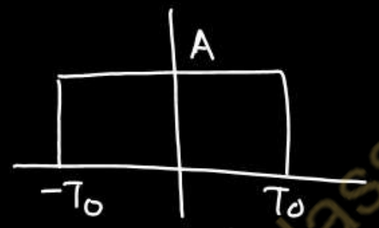
Sol. Fourier Transform

$$F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$= \int_{-T_0}^{T_0} A\, e^{-j\omega t}\, dt$$

$$= A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_0}^{T_0}$$

$$= \frac{2A}{\omega}\left[\frac{e^{-j\omega T_0} - e^{j\omega T_0}}{-2j}\right]$$



Gateway classes

Using Euler's identity,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Therefore,

$$\boxed{X(\omega) = \frac{2A}{\omega}\sin(\omega T_0)}$$

Expressing in Sinc Form

Since

$$\text{sinc}(x) = \frac{\sin x}{x},$$

multiply and divide by T_0:

$$X(\omega) = 2AT_0 \frac{\sin(\omega T_0)}{\omega T_0}$$

Hence,

$$\boxed{X(\omega) = 2AT_0\, \text{sinc}(\omega T_0)}$$

Ex. 9 Find the Fourier Transform of the signal

$F x(t) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$

$x(t) = e^{-2|t|}$ $\quad -\infty \to 0 \quad t \to -t \quad e^{-2(-t)} \quad e^{2t}$

Sol.

$$F\{x(t)\} = \int_{-\infty}^{0} e^{2t} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-2t} e^{-j\omega t}\, dt$$

Since, For $t < 0$, $|t| = -t$

For $t > 0$, $|t| = t$

$$F\{x(t)\} = \int_{-\infty}^{0} e^{t(2-j\omega)}\, dt + \int_{0}^{\infty} e^{-t(2+j\omega)}\, dt$$

$$= \left[\frac{e^{t(2-j\omega)}}{2-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-t(2+j\omega)}}{-(2+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{e^{0} - e^{-\infty}}{2-j\omega} + \frac{e^{-\infty} - e^{0}}{-(2+j\omega)}$$

$$= \frac{1}{2-j\omega} + \frac{1}{2+j\omega}$$

Simplification

$$X(\omega) = \frac{1}{2-j\omega} + \frac{1}{2+j\omega}$$

$$= \frac{(2+j\omega) + (2-j\omega)}{(2-j\omega)(2+j\omega)}$$

$$= \frac{4}{4+\omega^2}$$

Gateway classes

Ex. 10 Find the Fourier Transform of the signal (Periodic Signal)

x(t) = sin2t

$1 \rightarrow 2\pi\delta(\omega)$
$2\pi\delta(\omega-2)$
$2\pi\delta(\omega+2)$

Sol.

$$X(\omega) = \int_{-\infty}^{\infty} \sin 2t \; e^{-j\omega t}\, dt$$

Using Euler's identity,

$$\sin 2t = \frac{e^{j2t} - e^{-j2t}}{2j}$$

Substituting,

$$X(\omega) = \frac{1}{2j}\int_{-\infty}^{\infty} \left(e^{j2t} - e^{-j2t}\right) e^{-j\omega t}\, dt$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{\infty} e^{j(2-\omega)t}\, dt - \int_{-\infty}^{\infty} e^{-j(\omega+2)t}\, dt\right]$$

Using the Fourier Transform property,

$$\boxed{F\{1\} = 2\pi\delta(\omega)}$$

and

$$\boxed{F\{\sin\omega_0 t\} = \frac{1}{j}[\pi\delta(\omega-\omega_0) - \pi\delta(\omega+\omega_0)]}$$

$$\boxed{F\{\cos\omega_0 t\} = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)}$$

Therefore,

$$X(\omega) = \frac{1}{2j}[2\pi\delta(\omega-2) - 2\pi\delta(\omega+2)]$$

or,

$$\boxed{X(\omega) = \frac{\pi}{j}[\delta(\omega-2) - \delta(\omega+2)]}$$

Gateway classes

Ex. 11 Find the Fourier Transform of the signal

$$x(t) = 6\sin^2(4t)$$

Sol. Using the trigonometric identity,

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} = \frac{6}{2}[1-\cos(8t)] = 3[1-\cos(8t)]$$

Fourier Transform

$$F\{x(t)\} = 3F[1-\cos(8t)]$$

$$= 3[F\{1\} - F\{\cos(8t)\}]$$

$F\{1\} \rightarrow 2\pi\delta\omega$; $\cos(8t) \rightarrow 8$

Using the standard Fourier Transform pairs,

$$F\{1\} = 2\pi\delta(\omega)$$

$$F\{\cos(\omega_0 t)\} = \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

Where, $\omega_0 = 8$.

Substituting,

$$= 3[2\pi\delta(\omega) - \{\pi\delta(\omega-8) + \pi\delta(\omega+8)\}]$$

$$\boxed{X(\omega) = 6\pi\,\delta(\omega) - 3\pi\,\delta(\omega-8) - 3\pi\,\delta(\omega+8)}$$

$\sin\omega_0 t \rightarrow 2\pi[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$

$\cos\omega_0 t \rightarrow 2\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

Gateway classes

Ex. 12 Find the Fourier Transform of the signal Imp

$$X(j\omega) = \begin{cases} 2\cos\omega, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases}$$

Sol. Inverse Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cos\omega\; e^{j\omega t}\,d\omega$$

Using, $\cos\omega = \frac{e^{j\omega}+e^{-j\omega}}{2}$,

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{e^{j\omega}+e^{-j\omega}}{2} e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(e^{j\omega(t+1)} + e^{j\omega(t-1)}\right) d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(t+1)}}{j(t+1)} + \frac{e^{j\omega(t-1)}}{j(t-1)}\right]_{-\pi}^{\pi}$$

Substituting the limits,

$$= \frac{1}{2\pi}\left[\frac{e^{j\pi(t+1)} - e^{-j\pi(t+1)}}{j(t+1)} + \frac{e^{j\pi(t-1)} - e^{-j\pi(t-1)}}{j(t-1)}\right]$$

Using

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta,$$

$$= \frac{\sin\pi(t+1)}{\pi(t+1)} + \frac{\sin\pi(t-1)}{\pi(t-1)}$$

or, in **sinc** notation,

$$\boxed{x(t) = \text{sinc}(t+1) + \text{sinc}(t-1)}$$

Gateway classes

Ex.13 $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$ be an **LTI system**.

Find: (a) Transfer function $H(\omega) = \frac{Y(j\omega)}{X(j\omega)}$, **(b)** Output response for the input $x(t) = e^{-t}u(t)$

Sol.

(a) Find $H(\omega)$, Take Fourier Transform

Using

$$\mathcal{F}\left\{\frac{dy}{dt}\right\} = j\omega Y(j\omega), \quad \& \quad \mathcal{F}\left\{\frac{d^2y}{dt^2}\right\} = (j\omega)^2 Y(j\omega),$$

the differential equation becomes

$$(j\omega)^2 Y(j\omega) + 4(j\omega)Y(j\omega) + 3Y(j\omega) = (j\omega)X(j\omega) + 2X(j\omega)$$

$$Y(j\omega)[(j\omega)^2 + 4j\omega + 3] = X(j\omega)[j\omega + 2]$$

Hence,

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{2 + j\omega}{(j\omega)^2 + 4j\omega + 3}$$

Factor the denominator:

$$(j\omega)^2 + 4j\omega + 3 = (1 + j\omega)(3 + j\omega)$$

Therefore,

$$H(\omega) = \frac{2 + j\omega}{(1 + j\omega)(3 + j\omega)}$$

$e^{-at} \rightarrow \frac{1}{a+j\omega}$

Partial Fraction Expansion

$$H(\omega) = \frac{A}{1 + j\omega} + \frac{B}{3 + j\omega}$$

After solving,

$$A = \frac{1}{2}, B = \frac{1}{2}$$

Thus,

$$H(\omega) = \frac{1}{2(1 + j\omega)} + \frac{1}{2(3 + j\omega)}$$

$\frac{Y(\omega)}{X(\omega)}$ ⇐ $Y(\omega) = X(\omega) \cdot H(\omega)$

Impulse Response

Taking inverse Fourier transform,

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Gateway classes

(b) Output Response

Given

$$x(t) = e^{-t}u(t)$$

Its Fourier transform is

$$X(\omega) = \frac{1}{1+j\omega}$$

Using

$$Y(\omega) = H(\omega)X(\omega)$$

$$Y(\omega) = \underbrace{\frac{2+j\omega}{(1+j\omega)(3+j\omega)}}_{H(\omega)} \cdot \underbrace{\frac{1}{1+j\omega}}_{X(\omega)}$$

Hence,

$$Y(\omega) = \frac{2+j\omega}{(1+j\omega)^2(3+j\omega)}$$

Partial Fraction Expansion

$$Y(\omega) = \frac{A}{1+j\omega} + \frac{B}{(1+j\omega)^2} + \frac{C}{3+j\omega}$$

After solving,

$$A = \frac{1}{4}, B = \frac{1}{2}, C = -\frac{1}{4}$$

Thus,

$$Y(\omega) = \frac{1/4}{1+j\omega} + \frac{1/2}{(1+j\omega)^2} - \frac{1/4}{3+j\omega}$$

Inverse Fourier Transform

Using

$$\frac{1}{1+j\omega} \leftrightarrow e^{-t}u(t),$$

$$\frac{1}{(1+j\omega)^2} \leftrightarrow te^{-t}u(t),$$

$$\frac{1}{3+j\omega} \leftrightarrow e^{-3t}u(t),$$

the output is

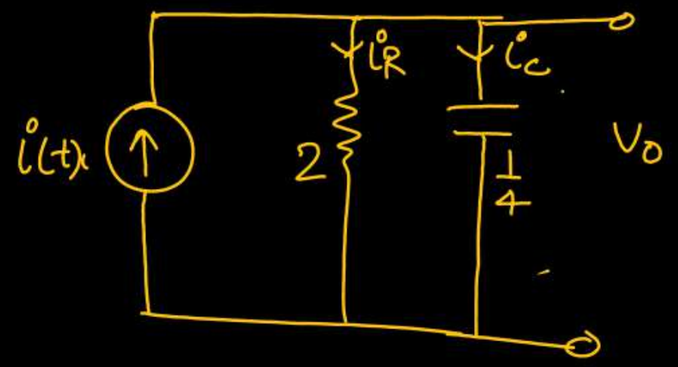
$$y(t) = \frac{1}{4}e^{-t}u(t) + \frac{1}{2}te^{-t}u(t) - \frac{1}{4}e^{-3t}u(t)$$

Gateway classes

Ex.14 For a given circuit current is given by $i(t) = 2e^{-t}u(t)$ determine output voltage of the circuit

Sol. For the parallel **R–C circuit**,

$$i(t) = i_R + i_C$$



Since

$$i_R = \frac{v_o(t)}{R}, i_C = C\frac{dv_o(t)}{dt}$$

equation becomes

$$i(t) = \frac{v_o(t)}{2} + \frac{1}{4}\frac{dv_o(t)}{dt} = 2e^{-t}u(t)$$

Take Fourier Transform

$$\frac{V_o(\omega)}{2} + \frac{1}{4} j\omega V_o(\omega) = \frac{2}{1+j\omega}$$

Using

$$2\left(\frac{1}{1+j\omega}\right) = \left(\frac{1}{2} + \frac{j\omega}{4}\right)V_o(\omega)$$

Simplifying,

$$\frac{2}{1+j\omega} = \frac{2+j\omega}{4}V_o(\omega)$$

Hence,

$$V_o(\omega) = \frac{4\times 2}{(2+j\omega)(1+j\omega)}$$

or

$$\boxed{V_o(\omega) = \frac{8}{(1+j\omega)(2+j\omega)}}$$

Partial Fraction Expansion

$$V_o(\omega) = \frac{A}{2+j\omega} + \frac{B}{1+j\omega}$$

After solving,

$$A = -8, B = 8$$

Therefore,

$$V_o(\omega) = \frac{8}{1+j\omega} - \frac{8}{2+j\omega}$$

Inverse Fourier Transform

Using

$$\frac{1}{a+j\omega} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} e^{-at}u(t),$$

we obtain

$$\boxed{v_o(t) = 8e^{-t}u(t) - 8e^{-2t}u(t)}$$

Gateway Classes

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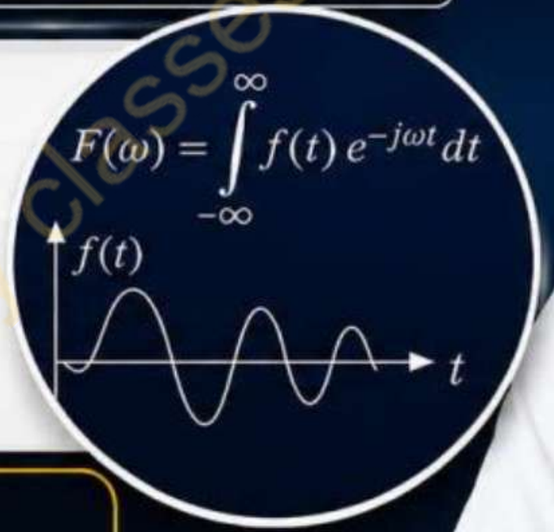
Fourier Transform



TOPICS TO BE COVERED :-

- Sampling Theorem

Avinash Sir

COMPLETE THEORY | IMPORTANT PROPERTIES | EXAM FOCUSED | HIGH SCORING TOPIC

Gateway Classes
GATEWAY CLASSES

Sampling Theorem

The **Sampling Theorem (Nyquist–Shannon Sampling Theorem)** states:

A continuous-time signal can be represented exactly by its samples and can be perfectly reconstructed from those samples **if the sampling frequency is greater than or equal to twice the highest frequency present in the signal.**

Mathematically,

$$f_s \geq 2f_m$$

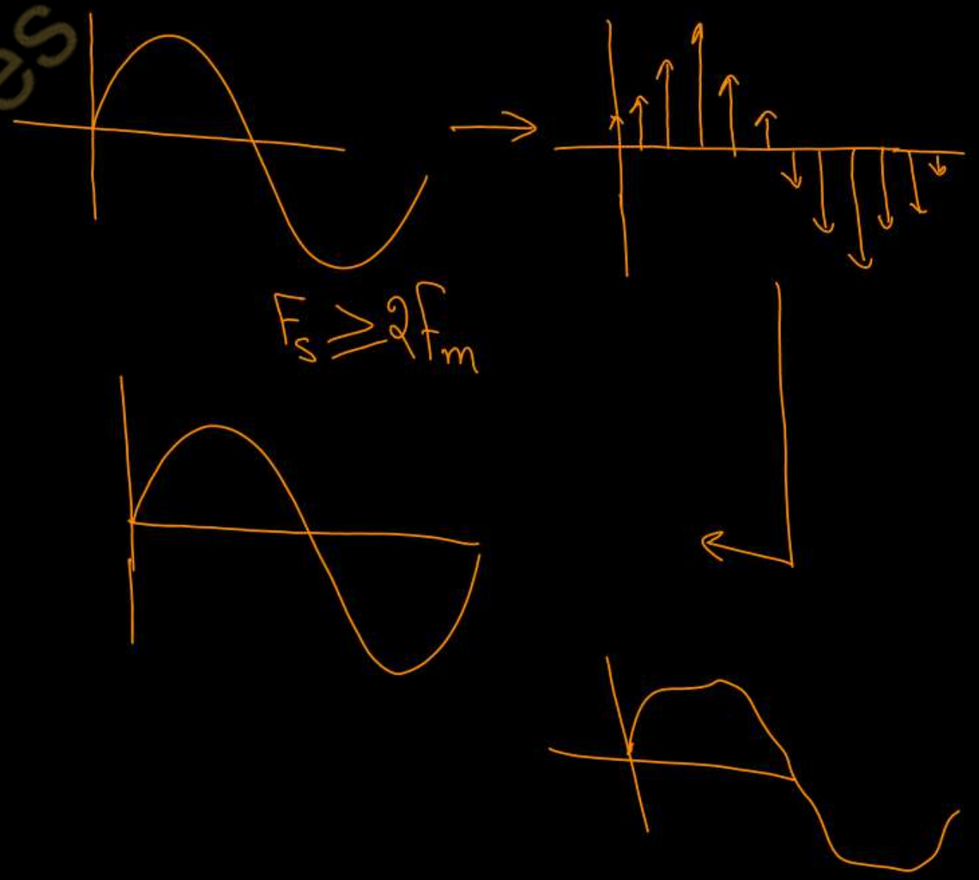
where:

- f_s = Sampling frequency (Hz)
- f_m = Highest frequency component of the message (analog) signal

The minimum sampling frequency,

$$f_N = 2f_m$$

is called the **Nyquist Rate**.



Gateway Classes

Sampling Theorem

Proof :=>

Let Message signal $x(t)$, Sampling (carrier) signal $\delta_T(t)$

The sampled signal is $\boxed{y(t) = x(t)\,\delta_T(t)}$

where $\delta_T(t)$ is a periodic impulse train given by

$$\boxed{\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)}$$

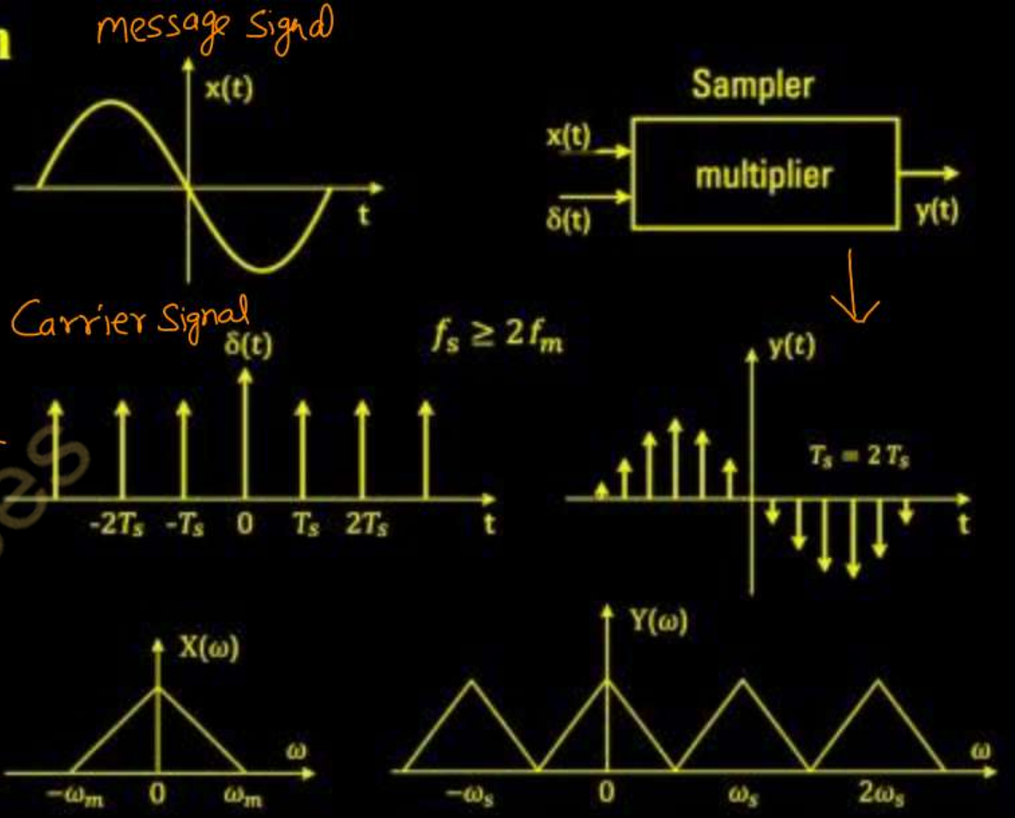
Where, T_s = Sampling period, $f_s = \frac{1}{T_s}$

The impulse train can be represented by the Fourier Series as

$$\delta_T(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_s t) + b_n \sin(n\omega_s t)]$$

Where, $\omega_s = 2\pi f_s$

Since the impulse train is an **even function**, $\boxed{b_n = 0}$ ($\because b_n = 0$)



Fourier Coefficients

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t)\,dt = \boxed{\frac{1}{T}}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} \delta_T(t)\cos(n\omega_s t)\,dt$$

Since, $\delta(0) = 1$, and, $\cos 0 = 1$,

$$\boxed{a_n = \frac{2}{T}}$$

Gateway classes

Sampling Theorem

Fourier Series Representation of the Impulse Train

For a periodic impulse train with sampling period T_s,

$$\delta_T(t) = \frac{1}{T_s} + \sum_{n=1}^{\infty} \frac{2}{T_s} \cos(n\omega_s t)$$

The sampled signal is

$$y(t) = x(t)\, \delta_T(t)$$

Substituting the Fourier series of the impulse train,

$$y(t) = x(t)\left[\frac{1}{T_s} + \frac{2}{T_s}\cos \omega_s t + \frac{2}{T_s}\cos 2\,\omega_s t + \cdots\right]$$

$$= \frac{x(t)}{T_s} + \frac{2}{T_s} x(t) \cos \omega_s t + \frac{2}{T_s} x(t) \cos 2\,\omega_s t + \cdots$$

Using the **Fourier Transform** modulation property,

$$\boxed{2x(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\leftrightarrow} X(\omega - \omega_0) + X(\omega + \omega_0)}$$

Fourier Transform of the Sampled Signal

Applying the Fourier transform term by term,

$$Y(\omega) = \frac{1}{T_s} X(\omega) + \frac{1}{T_s}[X(\omega + \omega_s) + X(\omega - \omega_s)] + \frac{1}{T_s}[X(\omega + 2\omega_s) + X(\omega - 2\omega_s)] + \cdots$$

The above expression can be written as

$$\boxed{Y(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)}$$

where

$$n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Thus, the original spectrum is repeated every

$$\boxed{\omega_s = 2\pi f_s}$$

Gateway classes

Sampling Theorem

Conditions of Sampling

1. Under Sampling (Aliasing): $f_s < 2f_m$

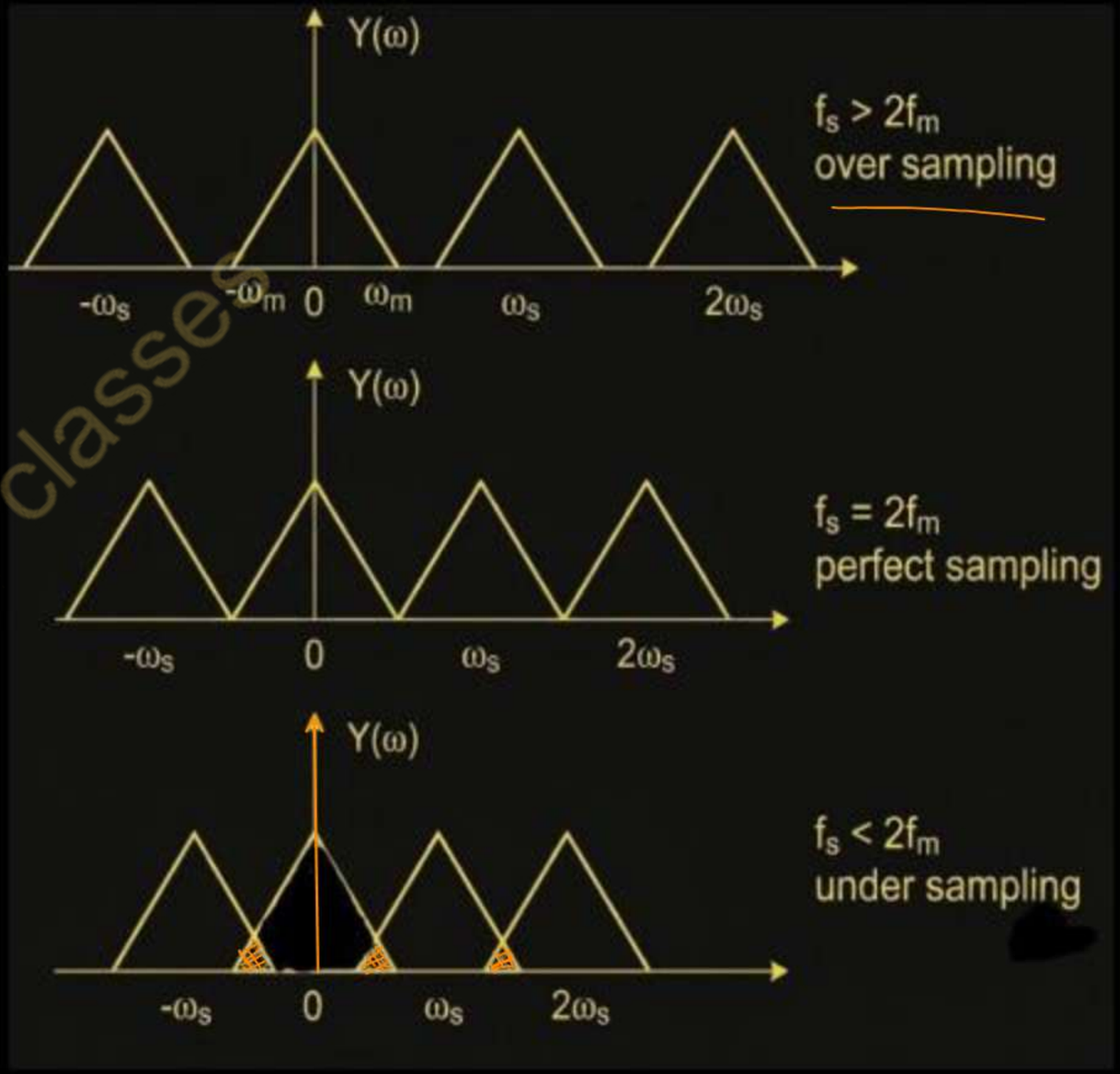
- Spectral overlap occurs.
- **Aliasing** is produced.
- The recovered signal becomes distorted.
- Original signal **cannot** be reconstructed.

2. Nyquist Sampling: $f_s = 2f_m$

- Sampling at the minimum permissible rate.
- Reconstruction is theoretically possible using an ideal low-pass filter.

3. Oversampling: $f_s > 2f_m$

- No spectral overlap.
- Perfect reconstruction is possible.
- Most practical systems use oversampling.



Gateway classes

Nyquist Rate : When the **sampling rate** becomes exactly equal to **twice the maximum frequency** present in the signal, it is called the **Nyquist Rate**. It is also known as the **minimum sampling rate** required for perfect reconstruction of a band-limited signal.

$$f_s = 2f_m$$

where: f_s= Sampling frequency (Hz) , f_m= Maximum frequency component of the message signal (Hz)

Nyquist Interval : The **maximum allowable sampling interval** corresponding to the Nyquist rate is called the **Nyquist Interval**.

$$T_s = \frac{1}{2f_m} \; seconds$$

where: T_s= Sampling interval (seconds)

Gateway classes

Nyquist Rate : When the **sampling rate** becomes exactly equal to **twice the maximum frequency** present in the signal, it is called the **Nyquist Rate**. It is also known as the **minimum sampling rate** required for perfect reconstruction of a band-limited signal.

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where: f_s= Sampling frequency (Hz) , f_m= Maximum frequency component of the message signal (Hz)

Nyquist Interval : The **maximum allowable sampling interval** corresponding to the Nyquist rate is called the **Nyquist Interval**.

$$T_s = \frac{1}{2f_m} \; seconds$$

where: T_s= Sampling interval (seconds)

Gateway classes

Find Nyquist rate and the Nyquist interval for the signal $x(t) = \frac{1}{2\pi}\cos(4000\pi t)\cos(1000\pi t)$

Solution

Given signal, $x(t) = \frac{1}{2\pi}\cos(4000\pi t)\cos(1000\pi t)$

Using the trigonometric identity,

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$x(t) = \frac{1}{4\pi}[2\cos(4000\pi t)\cos(1000\pi t)]$$

or

$$x(t) = \frac{1}{4\pi}[\cos(4000\pi t + 1000\pi t) + \cos(4000\pi t - 1000\pi t)]$$

$$\therefore\ x(t) = \frac{1}{4\pi}[\cos(5000\pi t) + \cos(3000\pi t)] \ldots (i)$$

Let the two frequencies present in the signal be ω_1 and ω_2. Then,

$$x(t) = \frac{1}{4\pi}[\cos\omega_1 t + \cos\omega_2 t] \ldots (ii)$$

Comparing equations (i) and (ii),

$$\omega_1 t = 2\pi f_1 t = 5000\pi t$$

$$2f_1 = 5000$$

$$\boxed{f_1 = 2500\ Hz}$$

Gateway classes

Second frequency $\omega_2 t = 2\pi f_2 t = 3000\pi t$

$$2f_2 = 3000$$

$$\boxed{f_2 = 1500\ Hz}$$

Therefore, the maximum frequency present in the signal is

$$\boxed{f_m = f_1 = 2500\ Hz}$$

Nyquist Rate: Nyquist rate is given by

$$f_s = 2f_m$$

where f_m is the maximum frequency present in the signal.

Here, $f_m = 2500\ Hz$

Therefore, $f_s = 2 \times 2500$

$$\boxed{f_s = 5000\ Hz = 5\ kHz}$$

Nyquist Rate = 5 kHz

Nyquist Interval

Nyquist interval is given by

$$T_s = \frac{1}{2f_m}$$

$$= \frac{1}{2 \times 2500}$$

$$= \frac{1}{5000}$$

$$= 0.2 \times 10^{-3}\ s$$

$$\boxed{T_s = 0.2\ ms = 200\ \mu s}$$

Answer: Nyquist Interval = 0.2 ms (200 µs)

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